



## Dimensional Analysis

**R Horan & M Lavelle**

The aim of this package is to provide a short self assessment programme for students who wish to learn how to use dimensional analysis to investigate scientific equations.

Copyright © 2003 [rhoran@plymouth.ac.uk](mailto:rhoran@plymouth.ac.uk) , [mlavelle@plymouth.ac.uk](mailto:mlavelle@plymouth.ac.uk)

Last Revision Date: June 2, 2004

Version 1.0

# Table of Contents

1. Introduction
2. Checking Equations
3. Dimensionless Quantities
4. Final Quiz
  - Solutions to Exercises
  - Solutions to Quizzes

The full range of these packages and some instructions, should they be required, can be obtained from our web page [Mathematics Support Materials](#).

# 1. Introduction

It is important to realise that it only makes sense to add the same sort of quantities, e.g. **area may be added to area** but **area may not be added to temperature!** These considerations lead to a powerful method to analyse scientific equations called **dimensional analysis**.

One should note that while **units are arbitrarily chosen** (an alien civilisation will not use seconds or weeks), **dimensions represent fundamental quantities** such as time.

Basic dimensions are written as follows:

<b>Dimension</b>	<b>Symbol</b>
Length	L
Time	T
Mass	M
Temperature	K
Electrical current	I

See the package on **Units** for a review of **SI units**.

**Example 1** An area can be expressed as a length times a length. Therefore the **dimensions of area** are  $L \times L = L^2$ . (A given area could be expressed in the SI units of square metres, or indeed in any appropriate units.) We sometimes write: **[area]** =  $L^2$

In some equations symbols appear which do **not** have any associated dimension, **e.g.**, in the formula for the area of a circle,  $\pi r^2$ ,  $\pi$  is just a **number** and does **not** have a dimension.

**EXERCISE 1.** Calculate the dimensions of the following quantities (click on the **green** letters for the solutions).

(a) Volume

(b) Speed

(c) Acceleration

(d) Density

**Quiz** Pick out the units that have a different dimension to the other three.

(a)  $\text{kg m}^2 \text{s}^{-2}$

(b)  $\text{g mm}^2 \text{s}^{-2}$

(c)  $\text{kg}^2 \text{m s}^{-2}$

(d)  $\text{mg cm}^2 \text{s}^{-2}$

## 2. Checking Equations

**Example 2** Consider the equation

$$y = x + \frac{1}{2}kx^3$$

Since any terms which are added together or subtracted **MUST** have the same dimensions, in this case  $y$ ,  $x$  and  $\frac{1}{2}kx^3$  have to have the same dimensions.

We say that such a scientific equation is **dimensionally correct**. (If it is not true, the equation must be wrong.)

If in the above equation  $x$  and  $y$  were both lengths (dimension L) and  $1/2$  is a dimensionless number, then for the  $\frac{1}{2}kx^3$  term to have the same dimension as the other two, we would need:

$$\begin{aligned}\text{dimension of } k \times L^3 &= L \\ \therefore \text{dimension of } k &= \frac{L}{L^3} = L^{-2}\end{aligned}$$

So  $k$  would have dimensions of **one over area**, i.e.,  $[k] = L^{-2}$ .

**Quiz** Hooke's law states that the force,  $F$ , in a spring extended by a length  $x$  is given by  $F = -kx$ . From Newton's second law  $F = ma$ , where  $m$  is the mass and  $a$  is the acceleration, calculate the dimension of the spring constant  $k$ .

- (a)  $\text{MT}^{-2}$       (b)  $\text{MT}^2$       (c)  $\text{ML}^{-2}\text{T}^{-2}$       (d)  $\text{ML}^2\text{T}^2$

**Example 3** The expressions for **kinetic energy**  $E = \frac{1}{2}mv^2$  (where  $m$  is the mass of the body and  $v$  is its speed) and **potential energy**  $E = mgh$  (where  $g$  is the acceleration due to gravity and  $h$  is the height of the body) look very different but both describe energy. One way to see this is to note that they have the same dimension.

Dimension of **kinetic energy**

$$\begin{aligned}\frac{1}{2}mv^2 &\Rightarrow \text{M}(\text{L}\text{T}^{-1})^2 \\ &= \text{ML}^2\text{T}^{-2}\end{aligned}$$

Dimension of **potential energy**

$$\begin{aligned}mgh &\Rightarrow \text{M}(\text{L}\text{T}^{-2})\text{L} \\ &= \text{ML}^2\text{T}^{-2}\end{aligned}$$

Both expressions have the same dimensions, they can therefore be added and subtracted from each other.

**EXERCISE 2.** Check that the dimensions of each side of the equations below agree (click on the **green** letters for the solutions).

(a) The volume of a cylinder of radius  $r$  and length  $h$

$$V = \pi r^2 h.$$

(c)  $E = mc^2$  where  $E$  is energy,  $m$  is mass and  $c$  is the speed of light.

(b)  $v = u + at$  for an object with initial speed  $u$ , (constant) acceleration  $a$  and final speed  $v$  after a time  $t$ .

(d)  $c = \lambda\nu$ , where  $c$  is the speed of light,  $\lambda$  is the wavelength and  $\nu$  is the frequency

**Note** that dimensional analysis is a way of checking that equations **might** be true. It does not prove that they are definitely correct. Dimensional analysis would suggest that both Einstein's equation  $E = mc^2$  and the (incorrect) equation  $E = \frac{1}{2}mc^2$  might be true. On the other hand dimensional analysis shows that  $E = mc^3$  makes no sense.

### 3. Dimensionless Quantities

Some quantities are said to be dimensionless. These are then pure numbers which would be the same no matter what units are used (e.g., the mass of a proton is roughly 1850 times the mass of an electron no matter how you measure mass).

**Example 4** The ratio of one mass  $m_1$  to another mass  $m_2$  is dimensionless:

$$\text{dimension of the fraction } \frac{m_1}{m_2} = \frac{\text{M}}{\text{M}} = 1$$

The dimensions have canceled and the result is a number (which is independent of the units, i.e., it would be the same whether the masses were measured in kilograms or tonnes).

**Note** that **angles** are defined in terms of ratios of lengths. They are therefore **dimensionless!** **Functions of dimensionless variables are themselves dimensionless.**



Very many functions are dimensionless. The following quantities are **important cases of dimensionless quantities**:

Trigonometric functions

Logarithms

Exponentials

Numbers, e.g.,  $\pi$

**Note** the following properties:

- Functions of dimensionless variables are dimensionless.
- Dimensionless functions must have dimensionless arguments.

**Quiz** If the number of radioactive atoms is found to be given as a function of time  $t$  by

$$N(t) = N_0 \exp(-kt)$$

where  $N_0$  is the number of atoms at time  $t = 0$ , what is the dimension of  $k$ ?

- (a) LT      (b) log(T)      (c) T      (d) T<sup>-1</sup>

**EXERCISE 3.** Determine the dimensions of the expressions below (click on the **green** letters for the solutions).

- (a) In a Young's slits experiment the angle  $\theta$  of constructive interference is related to the wavelength  $\lambda$  of the light, the spacing of the slits  $d$  and the order number  $n$  by  $d \sin(\theta) = n\lambda$ . Show that this is dimensionally correct.
- (b) The Boltzmann distribution in thermodynamics involves the factor  $\exp(-E/(kT))$  where  $E$  represents energy,  $T$  is the temperature and  $k$  is Boltzmann's constant. Find the dimensions of  $k$ .

**Quiz** Use dimensional analysis to see which of the following expressions is allowed if  $P$  is a pressure,  $t$  is a time,  $m$  is a mass,  $r$  is a distance,  $v$  is a velocity and  $T$  is a temperature.

- (a)  $\log\left(\frac{Pt}{mr}\right)$    (b)  $\log\left(\frac{Prt^2}{m}\right)$    (c)  $\log\left(\frac{Pr^2}{mt^2}\right)$    (d)  $\log\left(\frac{Pr}{mtT}\right)$

## 4. Final Quiz

**Begin Quiz** Choose the solutions from the options given.

- Newton's law of gravity states that the gravitational force between two masses,  $m_1$  and  $m_2$ , separated by a distance  $r$  is given by  $F = Gm_1m_2/r^2$ . What are the dimensions of  $G$ ?  
(a)  $L^3 M^{-1} T^{-2}$     (b)  $M^2 L^{-2}$     (c)  $MLT^{-2}$     (d)  $M^{-1} L^{-3} T^2$
- The coefficient of thermal expansion,  $\alpha$  of a metal bar of length  $\ell$  whose length expands by  $\Delta\ell$  when its temperature increases by  $\Delta T$  is given by  $\Delta\ell = \alpha\ell\Delta T$ . What are the dimensions of  $\alpha$ ?  
(a)  $K^{-1}$     (b)  $L^2T^{-1}$     (c)  $L^2T^{-1}$     (d)  $L^{-2}K^{-1}$
- The position of a mass at the end of a spring is found as a function of time to be  $A \sin(\omega t)$ . Select the dimensions of  $A$  and  $\omega$ .  
(a) L & T    (b) L & Dimensionless  
(c)  $\sin(L)$  &  $T^{-1}$     (d) L &  $T^{-1}$

**End Quiz**

## Solutions to Exercises

**Exercise 1(a)** A **volume** is given by multiplying three lengths together:

$$\begin{aligned}\text{Dimension of volume} &= L \times L \times L \\ &= L^3\end{aligned}$$

So **[volume]** =  $L^3$

(The SI units of volume are cubic metres.)

Click on the **green** square to return



**Exercise 1(b)** Speed is the rate of change of distance with respect to time.

$$\begin{aligned}\text{Dimensions of speed} &= \frac{\text{L}}{\text{T}} \\ &= \text{L T}^{-1}\end{aligned}$$

So  $[\text{speed}] = \text{L T}^{-1}$

(The SI units of speed are metres per second.)

Click on the **green** square to return



**Exercise 1(c)** **Acceleration** is the rate of change of speed with respect to time

$$\begin{aligned}\text{Dimensions of acceleration} &= \frac{L T^{-1}}{T} \\ &= L T^{-2}\end{aligned}$$

So **[acceleration]** =  $L T^{-2}$

(The SI units of acceleration are metres per second squared.)

Click on the **green** square to return



**Exercise 1(d)** **Density** is the mass per unit volume, so using the dimension of volume we get:

$$\begin{aligned}\text{Dimensions of volume} &= \frac{M}{L^3} \\ &= ML^{-3}\end{aligned}$$

So **[density]** =  $ML^{-3}$   
(The SI units of density are  $\text{kg m}^{-3}$ .)

Click on the **green** square to return



**Exercise 2(a)** We want to check the dimensions of  $V = \pi r^2 h$ . We know that the dimensions of volume are  $[\text{volume}] = \text{L}^3$ . The right hand side of the equation has dimensions:

$$\text{dimensions of } \pi r^2 h = \text{L}^2 \times \text{L} = \text{L}^3$$

So **both sides have the dimensions of volume.**

Click on the **green** square to return





**Exercise 2(b)** We want to check the dimensions of the equation  $v = u + at$ . Since  $v$  and  $u$  are both speeds, they have dimensions  $L T^{-1}$ . Therefore we only need to verify that  $at$  has this dimension. To see this consider:

$$[at] = (L T^{-2}) \times T = L T^{-2} \times T = L T^{-1}$$

So the equation is dimensionally correct, and all the terms have dimensions of speed.

Click on the **green** square to return



**Exercise 2(c)** We want to check the dimensions of the equation  $E = mc^2$ . Since  $E$  is an **energy it has dimensions  $ML^2T^{-2}$** . The right hand side of the equation can also be seen to have this dimension, if we recall that  $m$  is a mass and  $c$  is the speed of light (with dimension  $LT^{-1}$ ). Therefore

$$\begin{aligned}[mc^2] &= M(LT^{-1})^2 \\ &= ML^2T^{-2}\end{aligned}$$

So the equation is dimensionally correct, and all terms that we add have dimensions of energy.

Click on the **green** square to return



**Exercise 2(d)** We want to check the dimensions of the equation  $c = \lambda\nu$ . Since  $c$  is the **speed of light it has dimensions  $L T^{-1}$** . The right hand side of the equation involves wavelength  $[\lambda] = L$  and frequency  $[\nu] = T^{-1}$ . We thus have

$$[\lambda\nu] = L \times T^{-1}$$

which indeed also has dimensions of speed, so the equation is dimensionally correct.

Click on the **green** square to return



**Exercise 3(a)** We want to check the dimensions of  $d \sin(\theta) = n\lambda$ . Both  $d$  and  $\lambda$  have dimensions of **length**. The angle  $\theta$  and  $\sin(\theta)$  as well as the number  $n$  must all be dimensionless. Therefore we have

$$\begin{aligned}L \times 1 &= 1 \times L \\ \therefore [d \sin(\theta)] &= [n\lambda]\end{aligned}$$

So **both sides have dimensions of length**.

Click on the **green** square to return



**Exercise 3(b)** The factor  $\exp(-E/(kT))$  is an exponential and so must be dimensionless. Therefore its argument  $-E/(kT)$  must also be dimensionless.

The minus sign simply corresponds to multiplying by minus one and is dimensionless. Energy,  $E$  has dimensions  $[E] = \text{ML}^2 \text{T}^{-2}$  and temperature has dimensions  $\text{K}$ , so the dimensions of Boltzmann's constant are

$$[k] = \frac{\text{ML}^2 \text{T}^{-2}}{\text{K}} = \text{ML}^2 \text{T}^{-2} \text{K}^{-1}$$

(So the SI units of Boltzmann's constant are  $\text{kg m}^2 \text{s}^{-2} \text{K}^{-1}$ ).

Click on the **green** square to return



## Solutions to Quizzes

**Solution to Quiz:**  $\text{kg}^2 \text{m s}^{-2}$  has

$$\text{dimensions} = \text{M}^2 \text{L T}^{-2}$$

It can be checked that all the other answers have dimension  $\text{ML}^2 \text{T}^{-2}$ .

End Quiz

**Solution to Quiz:** From Hooke's law,  $F = -kx$ , we see that we can write

$$k = \frac{F}{x}$$

Now  $F = ma$ , so the **dimensions of force** are given by

$$\begin{aligned} [F] &= \text{M} \times (\text{L T}^{-2}) \\ &= \text{MLT}^{-2} \end{aligned}$$

Therefore the **spring constant** has dimensions

$$\begin{aligned} [k] &= \frac{\text{MLT}^{-2}}{\text{L}} \\ &= \text{MT}^{-2} \end{aligned}$$

End Quiz

**Solution to Quiz:** In  $N(t) = N_0 \exp(-kt)$ , the exponential and its argument must be dimensionless. Therefore  $kt$  has to be dimensionless. Thus

$$\text{dimensions of } k \times T = 1$$

So the dimension of  $k$  must be inverse time, i.e.,  $[k] = T^{-1}$ .

End Quiz



**Solution to Quiz:** First note that **the argument of a logarithm must be dimensionless**. Now pressure is force over area, so it has dimensions

$$\begin{aligned}[P] &= \frac{\text{MLT}^{-2}}{\text{L}^2} \\ &= \text{ML}^{-1}\text{T}^{-2}\end{aligned}$$

Therefore the combination  $\frac{Prt^2}{m}$  is dimensionless since

$$\begin{aligned}\left[\frac{Prt^2}{m}\right] &= \frac{(\text{ML}^{-1}\text{T}^{-2}) \times \text{L} \times \text{T}^2}{\text{M}} \\ &= \frac{\text{M}}{\text{M}} \\ &= 1\end{aligned}$$

None of the other combinations are dimensionless and so it would be completely meaningless to take their logarithms. **End Quiz**